Matched One-Pole Digital Shelving Filters

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1 Introduction

In digital audio processing, equalization is an enhancement or suppression of certain frequencies in order to compensate for spectral distortions in the transmission chain, or to account for room acoustics, or simply for personal preference. Filtering is often performed with recursive digital filters, aka IIR filters. Common designs of such filters often have an unwanted transfer function cramping towards high frequencies. In this note I present a design scheme for the specific class of first order shelving filters, with the objective to closely match the magnitude transfer curve of its analog counterpart.

2 Shelving Filters

Shelving filters have some gain factor G_0 for low frequencies, some other gain factor G_1 for high frequencies, and a transition region around some characteristic frequency f_c. For the sake of definiteness consider a high-shelf with $G_0 = 1$. Since G_1 is the only remaining gain parameter, we may omit the subscript and simply use G for the high shelf gain. In the analog domain, the transfer function is

$$H(s) = \frac{1 + \sqrt{G}s}{1 + s/\sqrt{G}}, \tag{1}$$

where $s = if/f_c$, $i = \sqrt{-1}$ and f is the frequency.

In the digital domain, the corresponding one-pole filter has a transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \tag{2}$$

where $z = \exp(i\pi f)$. We have chosen to denote the frequency f in units of the Nyquist frequency. Given the filter specification in terms of f_c and G, the objective is to find suitable filter coefficients a_1, b_0 and b_1 so that the magnitudes of the analog and the digital filter match.

3 Filter Design

Take the modulus squared of transfer function in equation (2),

$$|H(z)|^2 = \frac{1 + \beta\phi}{1 + \alpha\phi} \tag{3}$$

Here we have introduced

$$\phi = 1 - \cos(\pi f) \tag{4}$$

as a monotonic function which goes from zero at DC to two at the Nyquist frequency $f = 1$. Furthermore α and β are abreviations to keep the notation compact,

$$\begin{aligned} \alpha &= -2a_1/(1 + a_1)^2 \\ \beta &= -2b/(1 + b)^2, \quad b = b_1/b_0. \end{aligned} \tag{5}$$

Equation (3) is written in a form which satisfies the first of three requirements: (i) the digital transfer function is unity at DC,

$$b_0 + b_1 = 1 + a_1. \tag{6}$$

Two additional requirements determine all three coefficients a_1, b_0, b_1:
(ii) the transfer function matches the analog prototype at Nyquist frequency $f = 1$ or, equivalently, $\phi = 2$,

$$\frac{1 + 2\beta}{1 + 2\alpha} = \frac{1 + G/f_c^2}{1 + 1/(G f_c^2)} \tag{7}$$

and (iii) the transfer function matches the analog prototype at low frequencies to 2nd order. We may expand the magnitude squared of the analog and digital transfer functions, respectively, to obtain the following expressions.

$$\begin{aligned} |H(s)|^2 &= 1 + (G - 1/G) f^2/f_c^2 + \mathcal{O}(f^4) \\ |H(z)|^2 &= 1 + (\beta - \alpha)\pi^2 f^2/2 + \mathcal{O}(f^4) \end{aligned} \tag{8}$$

Equating the riht hand sides in equation (8) and using equation (7), we may solve for α and β,

$$\alpha = \frac{2}{\pi^2}\left(1 + \frac{1}{G f_c^2}\right) - \frac{1}{2}, \quad \beta = \frac{2}{\pi^2}\left(1 + \frac{G}{f_c^2}\right) - \frac{1}{2} \tag{9}$$

and from equation (5) we obtain

$$a_1 = \frac{-\alpha}{1 + \alpha + \sqrt{1 + 2\alpha}}, \quad b = \frac{-\beta}{1 + \beta + \sqrt{1 + 2\beta}}. \tag{10}$$

From the normalization condition for a high-shelf filter, equation (6), we finally get

$$b_0 = (1 + a)/(1 + b), \quad b_1 = b b_0 \tag{11}$$

Figure 2 shows a comparison of analog and digital transfer curves. The match is fairly good, certainly superior to the design of reference 1 (refer to figure 1).

A further improvement is to take the matching point in requirement (ii) slightly below the Nyquist frequency at $f_m = 0.9$. Equation (9) then becomes

$$\alpha = \frac{2}{\pi^2}\left(\frac{1}{f_m^2} + \frac{1}{G f_c^2}\right) - \frac{1}{\phi_m}, \quad \beta = \frac{2}{\pi^2}\left(\frac{1}{f_m^2} + \frac{G}{f_c^2}\right) - \frac{1}{\phi_m} \tag{12}$$

The rest of the equations remain valid.

4 Conclusion

Equations (10) and (12) provide closed-form expressions for the design of first order digital shelving filters. The resulting transfer magnitude functions match the analog prototype over the entire audio range, even for f_c above Nyquist.

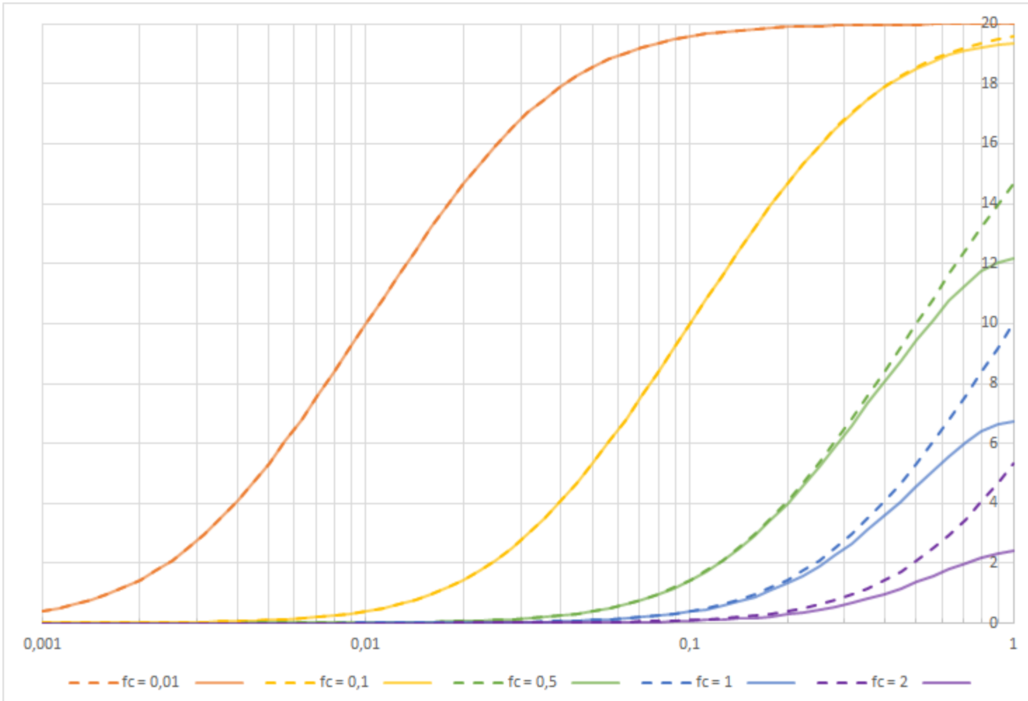

Figure 1: Magnitude response of various high-shelf filters. Solid lines: digital filter after [1]. Dashed lines: analog prototype. In all cases the high-shelf gain is 20 dB.

References

[1] Ch. Moore, First Order Digital Filters–An Audio Cookbook. Application note AN-11. http://freeverb3vst.osdn.jp/doc/AN11.pdf

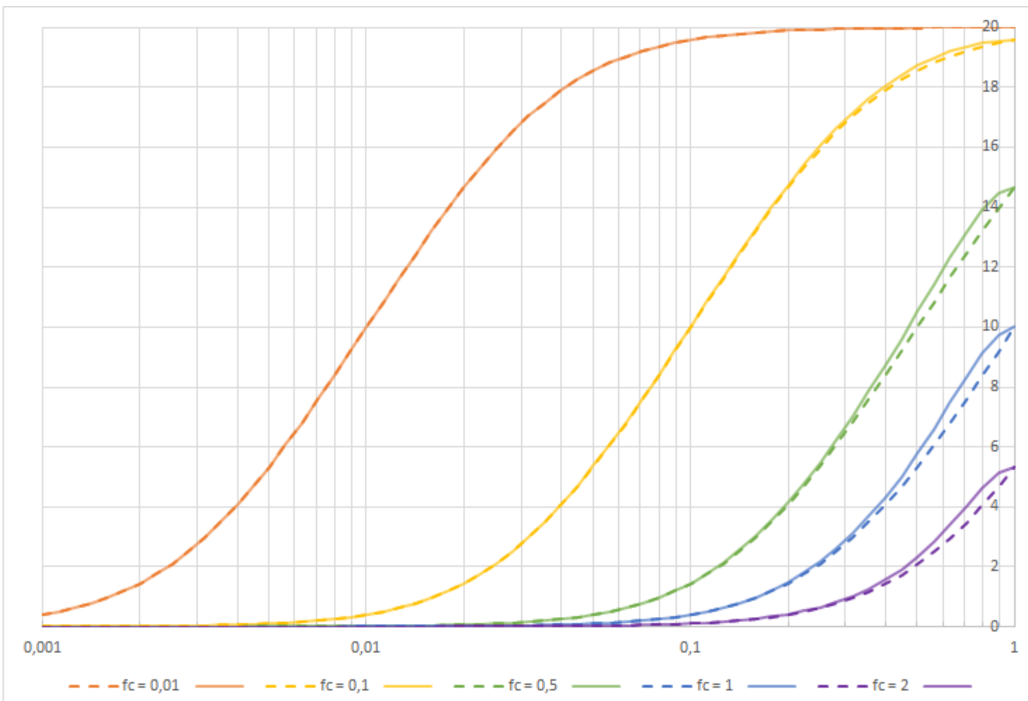

Figure 2: Magnitude response of various high-shelf filters. Solid lines: present work using equation (9). Dashed lines: analog prototype.

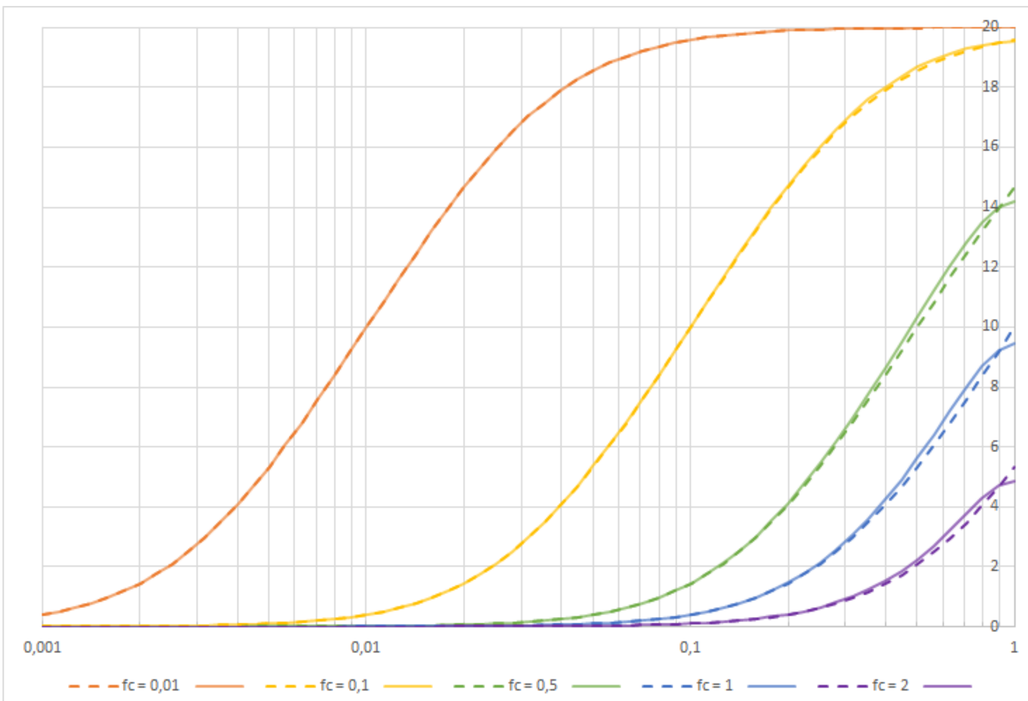

Figure 3: Magnitude response of various high-shelf filters. Solid lines: present work using equation (12). Dashed lines: analog prototype.