Waveshaper Harmonics

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Abstract

Waveshapers are often used to alter the harmonic structure of a given waveform. Three popular shapig functions are $\tanh(x)$, $x/\sqrt{1+x^2}$, and $\arctan(x)$. For these shaping functions we analyze the harmonic amplitudes of a shaped sine wave.

1 Introduction

Waveshaping [1] is a technique used in audio processing to alter the harmonic content and possibly also the dynamic range of a given signal. In the simplest case, a sine wave is transformed to a more complex waveform with some harmonic spectrum. Another application is in guitar distortion effects or, in a more subtle way, as saturation to emulate the imperfections of analog legacy equipment to add a vintage vibe. Nonlinear shaping is also used in devices called exciters [2] with the intent to restore lost spectral content in old recordings. It is also possible to *remove* spectral content e.g. by shaping a triangle wave to a sine wave.

Waveshaping functions may be of arbitrary form in principle. The most common ones display linear behavior for small signal amplitudes and then gradually saturate as the amplitude increases. However, in some synths [3] a particular timbre is achieved with a shaping function which folds back at some point.

In this note we investigate three more common shaping functions, the $\tanh(x)$ function, the algebraic $x/\sqrt{1+x^2}$-function, and the $\arctan(x)$-function, refer to figure 1. The particular tanh function is sometimes motivated by modeling semiconductor nonlinearities, the other two are merely some convenient expressions with the desired qualitative behavior.

2 The Tanh Shaper

Consider a cosine wave of amplitude a and circular frequency ω shaped by a tanh function. The result will be a periodic function with the same period as the incoming wave, however, due to the non-linearity there will be overtones with multiples of the fundamental frequency. The strength of these harmonics will depend on a, which is a measure of how hard we drive the tanh shaper. Because of symmetry of the tanh

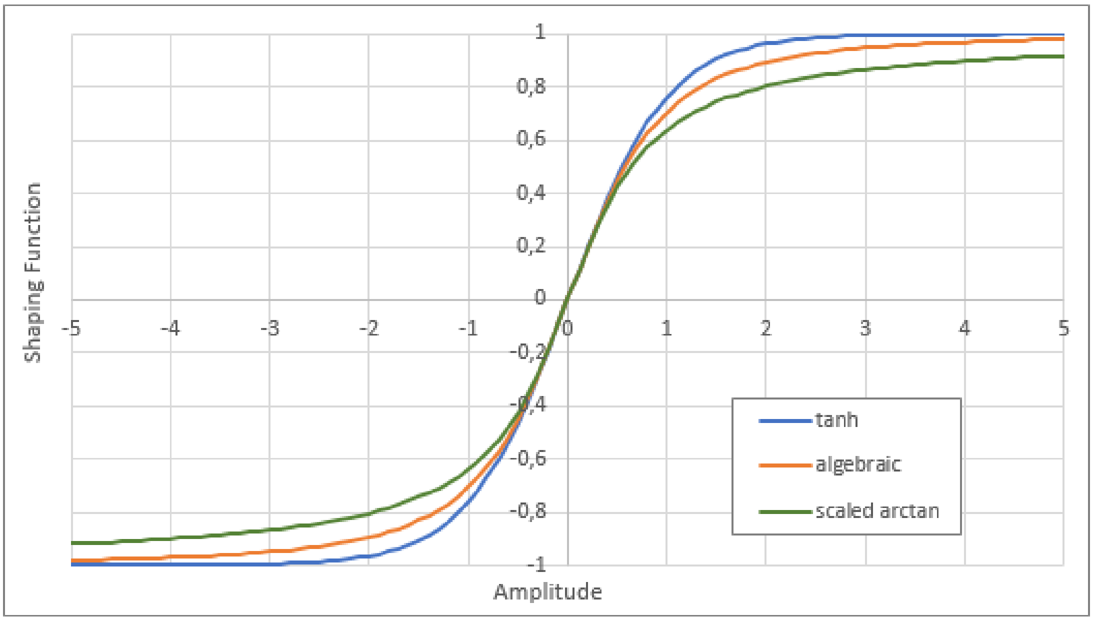

Figure 1: Three common shaping functions $\tanh(x)$, $x/\sqrt{1+x^2}$, and $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$.

function, only odd harmonics will be present, and only as cosine terms. We may thus expand the shaped wave into a Fourier series:

$$\tanh[a\cos(\omega t)] = \sum_{n=0}^{\infty} a_{2n+1}\cos((2n+1)\omega t). \tag{1}$$

The partial amplitudes a_{2n+1} may be computed from

$$a_{2n+1} = \frac{1}{\pi}\int_0^{2\pi} \tanh[a\cos(\phi)]\cos((2n+1)\phi)\,d\phi. \tag{2}$$

Unfortunately this integral cannot be computed in closed form, however we will consider limiting cases of small and large a, respectively, and derive a useful series representation for the general case. A fairly simple yet accurate approximation to the series is also given.

2.1 Limiting Cases

For small amplitudes $a < \pi/2$ the tanh function has a Taylor Series [4]

$$\tanh(x) = x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 \mp \ldots \tag{3}$$

Powers of cosines may be expressed in terms of cosines of multiple angles. If n is odd,

$$\cos(\phi)^n = \frac{2}{2^n}\sum_{k=0}^{(n-1)/2}\binom{n}{k}\cos((n-2k)\phi). \tag{4}$$

Hence, using the first three terms of the Taylor Series, we obtain

$$\begin{aligned} a_1 &= a - \tfrac{1}{4}a^3 + \tfrac{1}{12}a^5 \mp \dots \\ a_3 &= -\tfrac{1}{12}a^3 + \tfrac{1}{24}a^5 \mp \dots \\ a_5 &= \tfrac{1}{120}a^5 \mp \dots \end{aligned} \tag{5}$$

Observe that for small amplitude a, i.e a soft-driven shaper, the partial amplitudes fall off very rapidly with increasing n.

The other extreme of very large amplitudes a practically drives the shaper immediately into saturation (on both sides), resulting in a square wave output. The partial amplitudes of a square wave are given by

$$a_{2n+1} = \frac{(-1)^n\, 4}{(2n+1)\pi}, \quad a \gg n. \tag{6}$$

Obviously, a hard driven shaper generates much stronger harmonics extending to much higher frequencies.

2.2 The General Case

Now we turn to the general case of arbitrary amplitude a. The integral in eq.(2) may be evaluated by contour integration in the complex plane. We substitute $z = e^{i\phi}$, hence $d\phi = dz/iz$, and obtain

$$a_{2n+1} = \frac{1}{i\pi} \oint z^{2n} \tanh\left[a\,\frac{z+z^{-1}}{2}\right] dz, \tag{7}$$

where the integration is to be taken along the unit circle on the z-plane.

The function $\tanh(w)$ has isolated poles on the w-plane, hence we may expand into partial fractions [4],

$$\tanh(w) = -\sum_{k=0}^{\infty}\left[\frac{1}{w - i\pi(k+\frac{1}{2})} + \frac{1}{w + i\pi(k+\frac{1}{2})}\right]. \tag{8}$$

Substituting back $\frac{a}{2}(z+z^{-1})$ for w yields for each pole on the w-plane two poles on the z-plane, one inside and the other outside the unit circle. Collecting the residues associated with the poles inside the unit circle, we obtain

$$a_{2n+1} = \frac{(-1)^n 4}{a} \sum_{k=0}^{\infty} \frac{1}{s(r+s)^{2n+1}}, \tag{9}$$

with $r = (k+\frac{1}{2})\frac{\pi}{a}$ and $s = \sqrt{r^2+1}$. Figure 2 shows results for the first few partials for amplitudes in the range 0.1 to 100.

Equation (9) cannot, in general, be summed in closed form, however we can add the first few terms and approximate the remainder by an integral which, indeed, may be given in closed form.

$$a_{2n+1} \approx \frac{(-1)^n 4}{a}\left[\sum_{k=0}^{k_0} \frac{1}{s(r+s)^{2n+1}} + \frac{a}{(2n+1)\pi(r_0+s_0)^{2n+1}}\right], \tag{10}$$

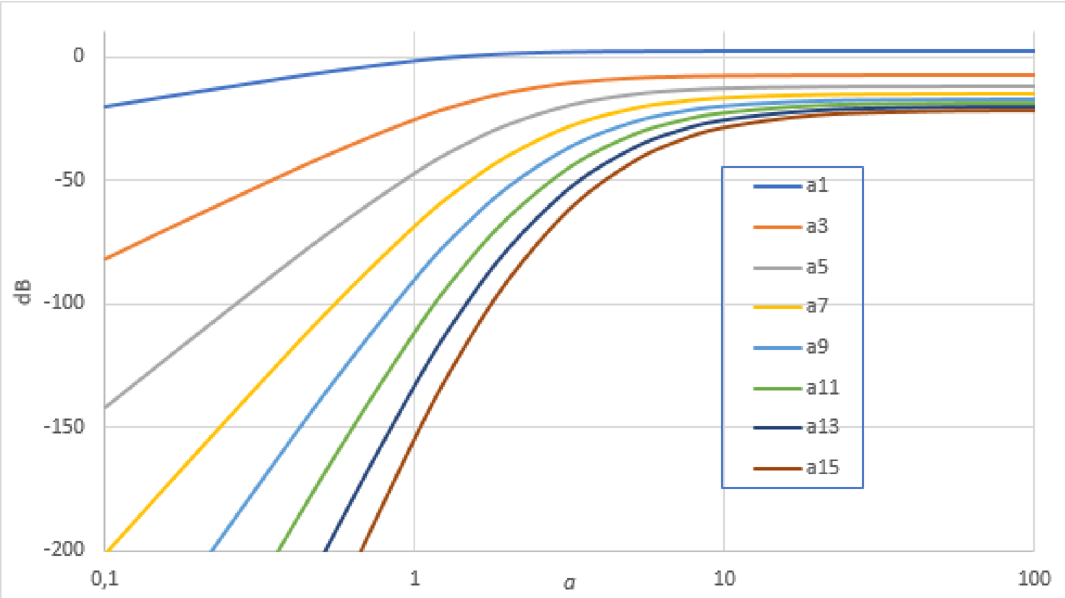

Figure 2: Partial amplitudes of a sine wave passed through a tanh waveshaper.

with $r_0 = (k_0 + 1)\frac{\pi}{a}$ and $s_0 = \sqrt{r_0^2 + 1}$. Eq.(10) is an accurate approximation to eq.(9). Even if we keep only the first term, i.e. $k_0 = 0$, the error is in the few percent range for all a and n. Indeed, if we drop even the first term, $k_0 = -1$, then $r_0 = 0$, $s_0 = 1$, and we immediately recover the large a limit in eq.(6).

The opposite limit of small a, eq.(5), is not so obvious but may be derived from eq.(9) by noting the following summation formulas [5],

$$\sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{2})^2} = \frac{\pi^2}{2}, \quad \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{2})^4} = \frac{\pi^4}{6}, \quad \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{2})^6} = \frac{\pi^6}{15}. \tag{11}$$

3 Algebraic Waveshaper

Consider a slightly different waveshaper function $x/\sqrt{1+x^2}$. A shaped wave $a\cos(\omega t)$ with amplitude a will have some Fourier series expansion

$$\frac{a\cos(\omega t)}{\sqrt{1 + a^2 \cos(\omega t)^2}} = \sum_{n=0}^{\infty} b_{2n+1} \cos((2n+1)\omega t). \tag{12}$$

The partial amplitudes b_{2n+1} may be computed from

$$b_{2n+1} = \frac{1}{\pi} \int_0^{2\pi} \frac{a\cos(\phi)}{\sqrt{1 + a^2\cos(\phi)^2}} \cos((2n+1)\phi)\, d\phi. \tag{13}$$

The first few of these integrals may be expressed in terms of elliptic functions, but that is not particularly helpful. As for the tanh waveshaper, we will first examine limiting cases of small and large amplitudes and then derive a general formula.

3.1 Limiting Cases

For small amplitudes $a < 1$ the algebraic waveshaper function has a Taylor Series [4]

$$\frac{x}{\sqrt{1+x^2}} = x - \tfrac{1}{2}x^3 + \tfrac{3}{8}x^5 \mp \ldots = \sum_{k=0}^{\infty} \beta_k x^{2k+1}. \tag{14}$$

The coefficients β_k may be computed recursively from

$$\beta_0 = 1, \quad \beta_{k+1} = -\frac{k+\frac{1}{2}}{k+1}\,\beta_k. \tag{15}$$

We will need this series again later on.

Using the first three terms of the Taylor Series and eq.(4), we obtain

$$\begin{aligned} b_1 &= a - \tfrac{3}{8}a^3 + \tfrac{15}{64}a^5 \mp \ldots \\ b_3 &= -\tfrac{1}{8}a^3 + \tfrac{15}{128}a^5 \mp \ldots \\ b_5 &= \tfrac{3}{128}a^5 \mp \ldots \end{aligned} \tag{16}$$

The expressions are of similar structure as eq.(5) for the tanh shaper, although the numbers are different.

The other extreme of very large amplitudes a is similar for the algebraic and the tanh shapers: in both cases, the result is a square wave with the same coefficients $a_{2n+1} = b_{2n+1}$ given in eq.(6).

3.2 The General Case

With the same substitution $z = e^{i\phi}$ as for the tanh shaper above, we obtain

$$b_{2n+1} = \frac{1}{i\pi} \oint \frac{dz}{\sqrt{z^2+v}} \frac{z^{2n+2}+z^{2n}}{\sqrt{v^{-1}+z^2}}, \tag{17}$$

with

$$v = \frac{a^2}{a^2+2+2\sqrt{a^2+1}}. \tag{18}$$

Observe that $0 \leq v < 1$. We may expand the first square root in eq.(17) in powers of vz^{-2},

$$\frac{1}{\sqrt{z^2+v}} = z^{-1} \sum_{k=0}^{\infty} \beta_k v^k z^{-2k} \tag{19}$$

and the second square root term in eq.(17) in powers of vz^2

$$\frac{1}{\sqrt{v^{-1}+z^2}} = v^{1/2} \sum_{\ell=0}^{\infty} \beta_\ell v^\ell z^{2\ell}. \tag{20}$$

In collecting all powers of z, note that only the z^{-1} terms contribute to the integral in eq.(17). We arrive at the final result

$$b_{2n+1} = 2v^{n+1/2} \sum_{k=0}^{\infty} \beta_k (\beta_{k+n} v^{2k} + \beta_{k+n+1} v^{2k+1}). \tag{21}$$

We may readily recover the small amplitude limit setting $v = a^2/4$ and keeping only the leading term,

$$b_{2n+1} \approx \frac{a^{2n+1}}{4^n}\beta_n, \quad a \ll 1. \tag{22}$$

The large amplitude limit is not so obvious, but has been confirmed numerically for the first few Fourier coefficients.

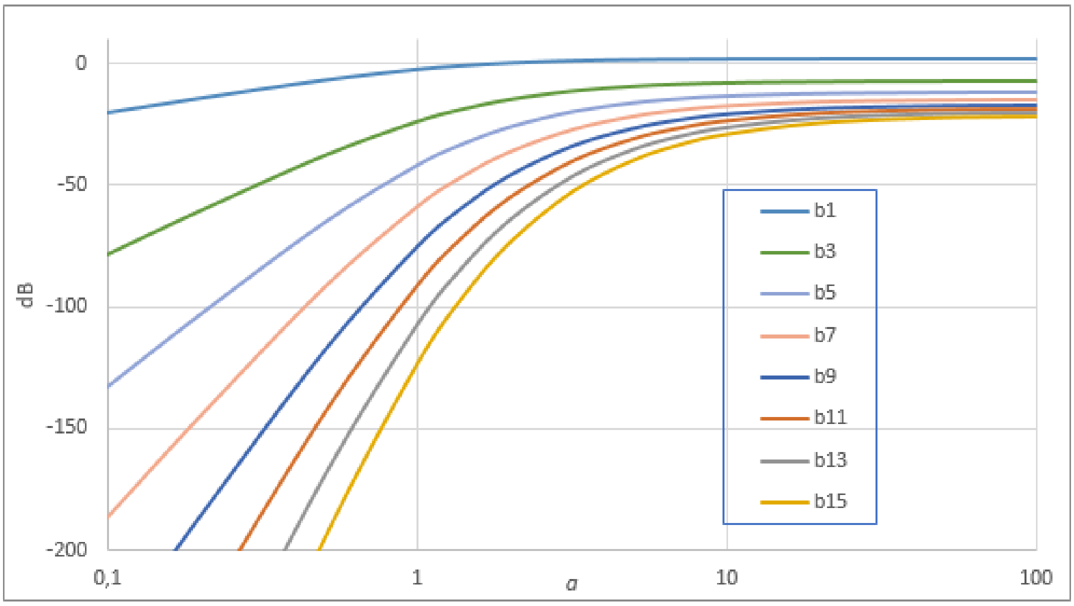

Figure 3: Partial amplitudes of a sine wave passed through a $x/\sqrt{1+x^2}$ waveshaper.

The sum in eq.(21) is a rather mild function of v in the relevant range from 0 to 1 and may be approximated by a straight line to within a few percent,

$$b_{2n+1} \approx 2\beta_n v^{n+1/2}(1-v) + \frac{(-1)^n 4}{(2n+1)\pi} v^{n+3/2}. \tag{23}$$

4 The Arctan Shaper

Another popular shaping function is $\arctan(x)$ or a scaled version $\frac{2}{\pi}\arctan(\frac{\pi}{2}x)$. A shaped wave $a\cos(\omega t)$ with amplitude a will have a Fourier series expansion

$$\arctan[a\cos(\omega t)] = \sum_{n=0}^{\infty} c_{2n+1}\cos((2n+1)\omega t). \tag{24}$$

The partial amplitudes c_{2n+1} may be computed from

$$c_{2n+1} = \frac{1}{\pi}\int_0^{2\pi} \arctan[a\cos(\phi)]\cos((2n+1)\phi)\,d\phi. \tag{25}$$

It turns out that this integral may be solved in closed form for all n. With the representation [4]

$$\arctan(z) = \int_0^z \frac{dt}{1+t^2} \tag{26}$$

we may write

$$c_{2n+1} = \frac{1}{\pi}\int_0^{2\pi} d\phi \int_0^a ds\, \frac{\cos(\phi)\,\cos((2n+1)\phi)}{1+s^2\cos(\phi)^2}. \tag{27}$$

Substituting $z = e^{i\phi}$ and interchanging the integration order we obtain

$$c_{2n+1} = \frac{2}{i\pi}\int_0^a \frac{ds}{s^2} \oint \frac{(z^2+1)\,z^{2n+1}\,dz}{(z^2+u)(z^2+u^{-1})}, \tag{28}$$

with

$$u = \frac{s^2}{s^2+2+2\sqrt{s^2+1}}. \tag{29}$$

Observe that $0 \le u < 1$. We may solve the inner integral by expanding into partial fractions. Two poles $z = \pm i\sqrt{u}$ lie inside the unit circle, hence the result is

$$\oint \frac{(z^2+1)\,z^{2n+1}\,dz}{(z^2+u)(z^2+u^{-1})} = 2\pi i\,\frac{(-1)^n u^{n+1}}{1+u}. \tag{30}$$

The remaining integral may be evaluated by solving eq.(29) for s and substituting

$$s = \frac{2\sqrt{u}}{1-u}, \quad ds = \frac{1+u}{(1-u)^2}\frac{du}{\sqrt{u}}. \tag{31}$$

Using eqs.(30) and (31), equation (28) becomes

$$c_{2n+1} = (-1)^n \int_0^v u^{n-1/2}\,du = (-1)^n\,\frac{v^{n+1/2}}{n+\frac{1}{2}}, \tag{32}$$

with v given in eq.(18). The simple result in eq.(32) suggests that perhaps there might be an easier derivation.

Figure 4 shows the first partial amplitudes as a function of input amplitude for the scaled arctan shaper.

From eq.(32) we may easily derive the small amplitude limit $a \ll 1$ by setting $v \approx \frac{a^2}{4} - \frac{a^4}{8}$,

$$\begin{aligned} c_1 &= a - \tfrac{1}{4}a^3 \pm \ldots \\ c_3 &= -\tfrac{1}{12}a^3 + \tfrac{1}{16}a^5 \mp \ldots \\ c_5 &= \tfrac{1}{80}a^5 - \tfrac{1}{64}a^7 \pm \ldots \end{aligned} \tag{33}$$

In figure 5 we compare the Fourier coefficients for the three shapers. The general behavior is very similar, although the onset of harmonics occurs already at lower amplitudes a for the soft arctan shaper compared to the hard tanh shaper. On the other hand, the increase is less steep, and the high-amplitude limit is attained more gradually. The curves for the algebraic shaper lie in between those for the other two. This is not unexpected from the shapes in figure 1.

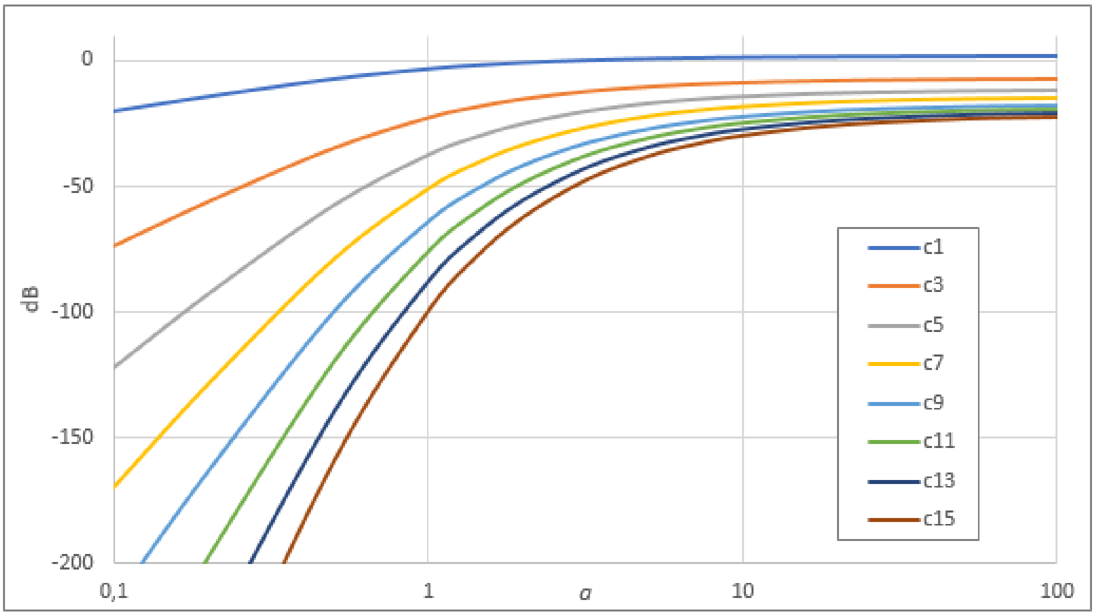

Figure 4: Partial amplitudes $|c_n|$ for the scaled arctan shaper.

5 Conclusion

We have analyzed the harmonic spectrum of a sine wave fed into a tanh, an algebraic and an arctan waveshaper, respectively. Simple expressions are given for the limiting cases of small ans large amplitudes, and series expansions as well as simple approximations for the general case. For the arctan shaper closed expressions are derived for the partial amplitudes.

Generally, the softness or hardness of the shape is reflected by the onset and growth of partials as a function of input amplitude a. Of the three shapers considered, tanh is the hardest ans arctan the softest shaper.

A plain sine wave is the simplest possible input signal, which results in an altered waveform with the same fundamental frequency and (odd) multiples thereof. For more than one sine wave, in addition to the formation of harmonics there will be inter-modulation giving rise to sums and differences of frequencies. The situation is even more complex for polyphonic input.

In guitar distortion effects, waveshaping is only one component in the signal processing chain. Pre and post filtering are very important for the result. It is largely subjective to say which of the shapers is more suited for guitar distortion, and it also depends on genre and the amount of distortion. Perhaps the soft arctan shaper is more useful for a crunchy blues guitar, whereas the tanh shaper might be more appropriate for heavily distorted metal powerchords as well as melodic solos with long sustain. The reader is encouraged to experiment and judge for herself.

The simple shaping model with a one-to-one correspondence of input and output amplitude at each instant may not be sufficient for a realistic emulation of nonlinear behavior in analog gear like valve amps or magnetic tape saturation. More sophisti-

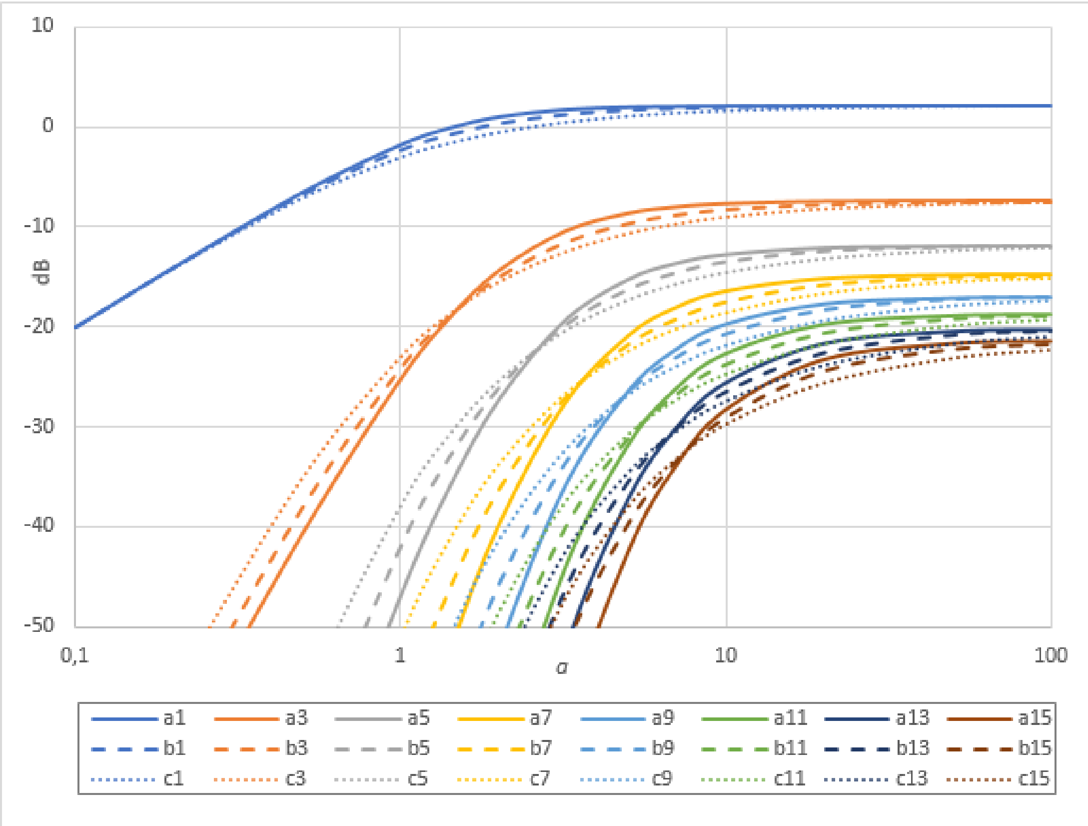

Figure 5: Comparison of partial amplitudes $|a_n|$, $|b_n|$, and $|c_n|$ for the tanh shaper, the algebraic shaper, and the scaled arctan shaper, respectively.

cated models include memory effects.

References

[1] https://en.wikipedia.org/wiki/Waveshaper

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